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CODIMENSION-2 HOPF BIFURCATION OF A TWO-DEGREE-OF-FREEDOM VIBRO-IMPACT SYSTEM

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Codimension-2 Hopf bifurcation problem of a two-degree-of-freedom system vibrating against a rigid surface is investigated in this paper. The four-dimensional Poincaré map of the vibro-impact system is reduced to a two-dimensional normal form by virtue of a center manifold reduction and a normal form technique. Then the theory of Hopf bifurcation of maps in R^2 is applied to conclude the existence of codimension-2 Hopf bifurcation of the vibro-impact system. The quasi-periodic response of the system by theoretical analysis is well supported by numerical simulations. It is shown that there exists codimension-2 Hopf bifurcation in multi-degree-of-freedom vibro-impact systems. The codimension-2 tori doubling phenomenon and the routes of quasi-periodic impacts to chaos are reported briefly.

1. INTRODUCTION

The optimum designs for a large number of mechanical systems with impacts, rely entirely upon an over-all knowledge of the dynamic mechanism of vibro-impact system. The phenomena of bifurcation and chaos in vibro-impact systems have been reported extensively in recent years in references [1–9], but few researchers have investigated the phenomena of Hopf bifurcation in vibro-impact systems. Xie [6] investigated the codimension-2 bifurcation and Hopf bifurcation of a single-degree-of-freedom system vibrating against an infinitely large plane. Ivanov [7] thought it possible for Hopf bifurcation to exist in multi-degree-of-freedom vibro-impact systems. In the recent past, Luo and Xie [8, 9] analyzed the existence of Hopf bifurcation in a two-degree-of-freedom vibro-impact system for the case of a single complex-conjugate pair of simple eigenvalues crossing the unit circle.

In this paper, we will analyze the existence of codimension-2 Hopf bifurcation of a two-degree-of-freedom vibro-impact system similar to the system considered by Luo and Xie [8, 9]. The rather lengthy procedure used for reducing the map of the system into a two-dimensional one by using a center manifold theorem in reference [10], is essentially similar to the one presented in great detail in reference [8]. It is reasonable to present only the corresponding abridgement and to focus on the derivation of codimension-2 normal form of the two-degree-of-freedom vibro-impact systems according to the theory of normal forms in references [11, 12]. The existence of codimension-2 Hopf bifurcation of the two-parameter normal form is more likely to be investigated by the theory of Hopf bifurcation of maps in R^2 in reference [13]. This theoretical solution is well verified by the results obtained by numerical simulations, which are represented by codimension-2 invariant circles in the projected Poincaré sections. It is shown that there exists

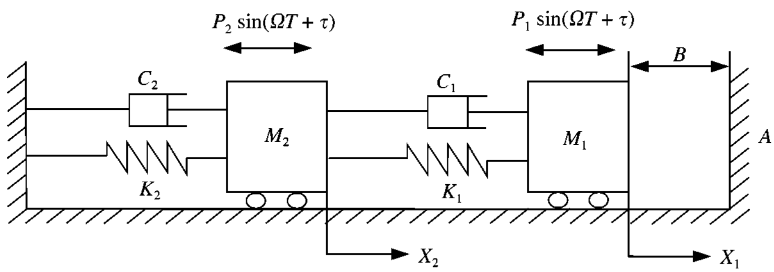

Figure 1. Schematic of the two-degree-of-freedom impact oscillator.

codimension-2 Hopf bifurcation in multi-degree-of-freedom vibro-impact systems under suitable system parameters. The codimension-2 tori doubling bifurcation and the routs of quasi-periodic impacts to chaos are also reported briefly.

2. ESTABLISHMENT AND REDUCTION OF POINCARE MAP

The mechanical model for a two-degree-of-freedom vibrator is shown in Figure 1. The mass M_1 impacts against a rigid surface when its displacement X_1 equals the gap B. The impact is described by a coefficient of restitution R.

Between impacts, for $X_1 < B$, the equations of motion of the two-degree-of-freedom impact oscillator with proportional damping are written in a non-dimensional form

$$\begin{bmatrix} 1 & 0 \\ 0 & \mu_m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\varsigma & -2\varsigma \\ -2\varsigma & 2\varsigma(1+\mu_c) \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1+\mu_k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1-f_2 \\ f_2 \end{Bmatrix}\sin(\omega t+\tau), \quad (x_1 < b) \tag{1}$$

and the impact equation of mass M_1 for $(x_1 = b)$ is

$$\dot{x}_{1+} = -R\dot{x}_{1-}. \quad (x_1 = b) \tag{2}$$

where the non-dimensional quantities are of the form

$$\mu_m = \frac{M_2}{M_1}, \quad \mu_k = \frac{K_2}{K_1}, \quad \mu_c = \mu_k, f_2 = \frac{P_2}{P_1+P_2}, \omega = \Omega\sqrt{\frac{M_1}{K_1}}, \varsigma = \frac{C_1}{2\sqrt{K_1 M_1}}, \tag{3a}$$

$$t = T\sqrt{\frac{K_1}{M_1}}, \quad b = \frac{BK_1}{P_1+P_2}, \quad x_i = \frac{X_i K_1}{P_1+P_2}, \quad \dot{x}_{1+} = \frac{\dot{X}_{1+}K_1}{P_1+P_2}, \quad \dot{x}_{1-} = \frac{\dot{X}_{1-}K_1}{P_1+P_2}. \tag{3b}$$

In equations (1) and (2), a dot denotes differentiation with respect to the non-dimensional time t, $\dot{x}_{1+}$ and $\dot{x}_{1-}$ represents the impacting mass velocities of approach and departure respectively. Let Ψ_i denote the canonical model matrix of equation (1) and take it as a transition matrix, under the change of variables

$$X = \Psi\xi. \tag{4}$$

Equation (1) reduces to

$$I\ddot{\xi} + C\dot{\xi} + \Lambda\xi = \bar{F}\sin(\omega t + \tau), \tag{5}$$

where $X = (x_1, x_2)^{\mathrm{T}}$, $\xi = (\xi_1, \xi_2)^{\mathrm{T}}$, I is a unit matrix 2×2, $C = 2\varsigma\Lambda = \mathrm{diag}[2\varsigma\omega_1^2, 2\varsigma\omega_2^2]$, $\Lambda = \mathrm{diag}[\omega_1^2, \omega_2^2]$, $\bar{F} = (\bar{f}_1, \bar{f}_2)^{\mathrm{T}} = \Psi^{\mathrm{T}} f$, $f = (1 - f_2, f_2)^{\mathrm{T}}$. It then follows that by using formal co-ordinate and modal matrix approach, the general solution of equation (1) takes the form

$$x_i = \sum_{j=1}^{2} \psi_{ij}(\mathrm{e}^{-\eta_j t}(a_j\cos\omega_{dj}t + b_j\sin\omega_{dj}t) + A_j\sin(\omega t + \tau) + B_j\cos(\omega t + \tau)), \tag{6a}$$

$$\dot{x}_i = \sum_{j=1}^{2} \psi_{ij}(\mathrm{e}^{-\eta_j t}((b_j\omega_{dj} - \eta_j a_j)\cos\omega_{dj}t - (a_j\omega_{dj} + \eta_j b_j)\sin\omega_{dj}t)$$

$$+ A_j\omega\cos(\omega t + \tau) - B_j\omega\sin(\omega t + \tau)) \quad (i, j = 1, 2), \tag{6b}$$

where Ψ_{ij} are elements of the canonical modal matrix Ψ, $\eta_j = \varsigma\omega_j^2$, $\omega_{dj} = \sqrt{\omega_j^2 - \eta_j^2}$, a_j and b_j are the constants of integration which are determined by the initial condition and modal parameters of the system, and A_j, B_j are the amplitude parameters taking the form

$$A_j = \frac{1}{2\omega_{dj}}\left(\frac{\omega + \omega_{dj}}{(\omega + \omega_{dj})^2 + \eta_j^2} - \frac{\omega - \omega_{dj}}{(\omega - \omega_{dj})^2 + \eta_j^2}\right)\bar{f}_j, \tag{7a}$$

$$B_j = \frac{\eta_j}{2\omega_{dj}}\left(\frac{1}{(\omega - \omega_{dj})^2 + \eta_j^2} - \frac{1}{(\omega + \omega_{dj})^2 + \eta_j^2}\right)\bar{f}_j. \tag{7b}$$

We can choose a Poincaré section $\sigma \subset R^4 \times S$, where $\sigma = (x_1, \dot{x}_1, x_2, \dot{x}_2, \theta) \in R^4 \times S$, $x_1 = b$, then the two-parameter Poincaré map of the system can be established as [8]

$$\Delta X' = f(v, b; \Delta X), \tag{8}$$

where $\Delta X \in R^4$, $\Delta X = (\Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\tau)^{\mathrm{T}}$, $\Delta X' = (\Delta\dot{x}'_{1+}, \Delta x'_{20}, \Delta\dot{x}'_{20}, \Delta\tau')^{\mathrm{T}}$ are the disturbed vectors in the hyperplane σ, $v = \omega/\omega_{d1}$ and b are real parameters, and $v \in R^1$, $b \in R^1$, such that $\Delta X^* = (0,0,0,0)^{\mathrm{T}}$ is a fixed point for system (8) in some neighborhood of the critical parameter values $v = v_c$, $b = b_c$, at which $\mathrm{D}f_{\Delta X}$ $(v, b; \Delta X^*)$ has double eigenvalues [14] $\lambda_{1,2}(v_c, b_c) = -1$, the other simple eigenvalues $\lambda_3(v_c, b_c)$, $\lambda_4(v_c, b_c)$ stay inside the unit circle, and $\mathrm{D}f_{\Delta X}(v_c, b_c); \Delta X^*)$ has the Jordan form

$$\begin{bmatrix} -1 & 1 & \\ & -1 & \\ & & D \end{bmatrix}, \tag{9}$$

where D is a real matrix with eigenvalues $\lambda_3(v_c, b_c)$ and $\lambda_4(v_c, b_c)$.

The space R^4 is then decomposed as follows

$$R^4 = E_O + E_-, \tag{10}$$

where E_O, E_- are eigenspaces commuting with $\lambda_{1,2}(v, b)$ or $\lambda_{3,4}(v, b)$ respectively.

Taking $\mu_1 = v - v_c$, $\mu_2 = b - b_c$, $\mu = [\mu_1, \mu_2]^{\mathrm{T}}$, for the map (8), there exists a local center manifold [10] Ξ_μ, on which the local behavior of the four-dimensional map (8) can be reduced into a two-dimensional map [8]. The two-dimensional map is now

$$Y' = F(\mu, Y), \tag{11}$$

where $Y \in R^2$, $F(\mu, 0) = 0$. After the change of variables

$$X = G(\mu, Y) = Y + \Phi(\mu, Y), \quad \Phi(\mu, 0) = 0, \quad X \in R^2, \tag{12}$$

We can reduce equation (11) into the following normal form according to the theory of normal forms in references [11, 12],

$$X' = H(\mu, X) = A_0 X + N(\mu, X), \tag{13}$$

where

$$A_0 = \begin{bmatrix} -1 & 1 \\ & -1 \end{bmatrix}, \quad N(\mu, 0) = 0.$$

In order to facilitate applications of our theoretical results, we give an explicit procedure using the derivation of codimension-2 normal form (13), as will be seen

We expand the functions $F(\mu, Y)$, $\Phi(\mu, Y)$, $N(\mu, X)$ as Taylor series with respect to μ, Y or X:

$$F(\mu, Y) = \sum_{p+q \geqslant 1} F_{pq}[\mu^{(p)}, Y^{(q)}], \tag{14a}$$

$$\Phi(\mu, Y) = \sum_{p+q \geqslant 1} \Phi_{pq}[\mu^{(p)}, Y^{(q)}], \tag{14b}$$

$$N(\mu, X) = \sum_{p+q \geqslant 1} N_{pq}[\mu^{(p)}, X^{(q)}], \tag{14c}$$

where

$$F_{pq} = \frac{1}{p!q!} \frac{\partial^{p+q} F}{\partial \mu^p \partial Y^q}(0, 0),$$

$$\Phi_{pq} = \frac{1}{p!q!} \frac{\partial^{p+q} \Phi}{\partial \mu^p \partial Y^q}(0, 0),$$

$$N_{pq} = \frac{1}{p!q!} \frac{\partial^{p+q} N}{\partial \mu^p \partial X^q}(0, 0).$$

From $H \circ G = G \circ F$, we obtain

$$\begin{aligned}
&A_0 \Phi_{11}[\mu, X] - \Phi_{11}[\mu, A_0 X] = F_{11}[\mu, X] - N_{11}[\mu, X], \\
&A_0 \Phi_{02}[X^{(2)}] - \Phi_{02}[(A_0 X)^{(2)}] = F_{02}[X^{(2)}] - N_{02}[X^{(2)}], \\
&A_0 \Phi_{03}[X^{(3)}] - \Phi_{03}[(A_0 X)^{(3)}] = F_{03}[X^{(3)}] + 2\Phi_{02}[A_0 X, F_{02}[X^{(2)}]] \\
&- N_{03}[X^{(3)}] - 2N_{02}[X, \Phi_{02}[X^{(2)}]], \\
&\vdots \\
&A_0 \Phi_{pq}[\mu^{(p)}, X^{(q)}] - \Phi_{pq}[\mu^{(p)}, (A_0 X)^{(q)}] = R_{pq}, \\
&\vdots
\end{aligned} \tag{15}$$

where R_{pq} is a combination of F_{nm}, N_{nm}, $N_{n'm'}$, and $\Phi_{n'm'}$ ($n + m \leqslant p + q$, $n' + m' \leqslant p + q - 1$). F_{pq}, Φ_{pq} and N_{pq} denote, respectively, the terms of order p of μ and order q of X, such as

$$\Phi_{12}[\mu, X^{(2)}] = \begin{bmatrix} (a_1\mu_1 + b_1\mu_2)x_1^2 + (c_1\mu_1 + d_1\mu_2)x_1x_2 + (e_1\mu_1 + f_1\mu_2)x_2^2, \\ (a_2\mu_1 + b_2\mu_2)x_1^2 + (c_2\mu_1 + d_2\mu_2)x_1x_2 + (e_2\mu_1 + f_2\mu_2)x_2^2 \end{bmatrix}.$$

Let $B_q\Phi_{pq}[\mu^{(p)}, X^{(q)}] = A_0\Phi_{pq}[\mu^{(p)}, X^{(q)}] - \Phi_{pq}[\mu^{(p)}, (A_0X)^{(q)}]$, and H_q denotes a space of homogeneous vector polynomials of degree q [15]. Taking $e_1 = [x_1, 0]^{\mathrm{T}}$, $e_2 = [x_2, 0]^{\mathrm{T}}$, $e_3 = [0, x_1]^{\mathrm{T}}$, $e_4 = [0, x_2]^{\mathrm{T}}$ as a basis in H_1, one calculates directly from the first equation of equations (15) that

$$B_1e_1 = -e_2, \quad B_1e_2 = 0, \quad B_1e_3 = e_1 - e_4, \quad B_1e_4 = e_2, \tag{16}$$

i.e., for the basis $\{e_1, e_2, e_3, e_4\}$, B_1 is written as

$$B_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{17}$$

In what follows, we write $\Phi_{11} = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]^{\mathrm{T}}$, $N_{11} = [n_1, n_2, n_3, n_4]^{\mathrm{T}}$, $F_{11} = [\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4]^{\mathrm{T}}$ and obtain

$$B_1\Phi_{11} = F_{11} - N_{11}. \tag{18}$$

Equations (18) has two solutions as

$$\Phi_{11} = [0, c, -\bar{f}_4, \bar{f}_2]^{\mathrm{T}}, \quad N_{11} = [\bar{f}_1 + \bar{f}_4, 0, \bar{f}_3, 0]^{\mathrm{T}} \tag{19a}$$

or

$$\Phi_{11} = [0, 0, \bar{f}_1, \bar{f}_4]^{\mathrm{T}}, \quad N_{11} = [0, 0, \bar{f}_3, \bar{f}_1 + \bar{f}_4]^{\mathrm{T}} \tag{19b}$$

where c is an arbitrary constant.

Similarly, taking $e_1 = [x_1^2, 0]^{\mathrm{T}}$, $e_2 = [x_1x_2, 0]^{\mathrm{T}}$, $e_3 = [x_2^2, 0]^{\mathrm{T}}$, $e_4 = [0, x_1^2]^{\mathrm{T}}$, $e_5 = [0, x_1x_2]^{\mathrm{T}}$, $e_6 = [0, x_2^2]^{\mathrm{T}}$ as a basis in H_2, one can easily show that

$$B_2 = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}. \tag{20}$$

Since B_2 is reversible, it follows from the second equation of equations (15) that

$$\Phi_{02} = B_2^{-1}F_{02}, \qquad N_{02}[X^{(2)}] = 0. \tag{21}$$

Repeating once more the same procedure above, we can find the two solutions of the third equation of equations (15):

$$\Phi_{03} = [\varphi_{31}, \varphi_{32}, \varphi_{33}, 0, 0, \bar{f}_{37} + 2\bar{f}_{38}, \bar{f}_{37} + 3\bar{f}_{38}, 0],$$

$$N_{03} = [\bar{f}_{31}, 0, 0, 0, \bar{f}_{35}, 0, 0, 0] \tag{22a}$$

or

$$\Phi_{03} = [\varphi_{31}, \varphi_{32}, \varphi_{33}, 0, \bar{f}_{31}, \bar{f}_{37} + 2\bar{f}_{38}, \bar{f}_{37} + 3\bar{f}_{38}, 0],$$

$$N_{03} = [0, 0, 0, 0, \bar{f}_{35}, 3\bar{f}_{31} + \bar{f}_{36}, 0, 0] \tag{22b}$$

where $\varphi_{3i}, \bar{f}_{3i}$ denote, respectively, the ith element of Φ_{03} and F_{03} for a suitable basis in H_3, and

$$\varphi_{31} = \tfrac{1}{3}(-\bar{f}_{32} + \bar{f}_{37} + 2\bar{f}_{38}),$$

$$\varphi_{32} = \tfrac{1}{2}(-\bar{f}_{32} - \bar{f}_{33} + 2\bar{f}_{37} + 5\bar{f}_{38}),$$

$$\varphi_{33} = \tfrac{1}{6}(-\bar{f}_{32} - 3\bar{f}_{33} - 6\bar{f}_{34} + 4\bar{f}_{37} + 11\bar{f}_{38}).$$

The aforementioned steps may be applied to solve the other equations in equation (15).

Let $\varepsilon_1 = \bar{f}_1 + \bar{f}_4$, $\varepsilon_2 = \bar{f}_3$ for (19a), or $\varepsilon_1 = \bar{f}_3$, $\varepsilon_2 = \bar{f}_1 + \bar{f}_4$ for (19b), $a_1 = 3\bar{f}_{31} + \bar{f}_{36}$, $a_2 = \bar{f}_{35}$, $a_3 = \bar{f}_{31}$. From equations (19), (21) and (22), we can obtain four equivalence normal forms. One of them is in the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 x_1 \\ \varepsilon_2 x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ a_1 x_1^2 x_2 + a_2 x_1^3 \end{pmatrix} + \begin{pmatrix} 0 \\ O((|x_1| + |x_2|)^5) \end{pmatrix}. \tag{23}$$

For convenience of notation, equation (23) is now written as

$$X' = f(\varepsilon_1, \varepsilon_2; X) \tag{24}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2, \ \varepsilon_1 = \varepsilon_1(\mu) \text{ and } \varepsilon_2 = \varepsilon_2(\mu)$$

are bifurcation parameters, a_1 and a_2 are constants.

It follows that the period two points of equation (24) satisfy

$$f^2(\varepsilon_1, \varepsilon_2; X) = X. \tag{25}$$

Ignoring the terms of high order, the solutions of equation (25) are then

$$X1 = \left(\sqrt{\frac{\varepsilon_2}{-a_2}}, \quad -\varepsilon_1 \sqrt{\frac{\varepsilon_2}{-a_2}} \right), \tag{26a}$$

$$X2 = \left(-\sqrt{\frac{\varepsilon_2}{-a_2}}, \quad \varepsilon_1 \sqrt{\frac{\varepsilon_2}{-a_2}} \right), \tag{26b}$$

The Jacobian operator A_2 of map $f^2(\varepsilon_1, \varepsilon_2; X)$ at the period two points $X1, X2$ has the form

$$A_2 = \mathrm{D}f^2(\varepsilon_1, \varepsilon_2;\, Xi) = \begin{bmatrix} 1 - 2(\varepsilon_1 + \varepsilon_2) & -2 + \varepsilon_1 - \dfrac{a_1}{a_2}\varepsilon_2 \\ 4\varepsilon_2 & 1 + 2\left(-1 + \dfrac{a_1}{a_2}\right)\varepsilon_2 \end{bmatrix}, \quad i = 1, 2 \tag{27}$$

and

$$\mathrm{tr}\, A_2 = 2\left(1 - \varepsilon_1 + \left(\frac{a_1}{a_2} - 2\right)\varepsilon_2\right),\ \det A_2 = 1 - 2\varepsilon_1 + 2\left(\frac{a_1}{a_2} + 2\right)\varepsilon_2. \tag{28}$$

When $\varepsilon_2 = \varepsilon_2^* = 1/(2 + a_1/a_2)\varepsilon_1$, it is clear that $\det A_2 = 1$.

Let $\varepsilon = 2(a_1/a_2 + 2)\varepsilon_2 - 2\varepsilon_1$, then ε_2, passing through ε_2^* while ε_1 is fixed on a sufficiently small value ε_1^*, corresponds to ε crossing 0. Because of the center manifold [10] Ξ_μ the local dynamic behavior of the four-dimensional map (8) can be reduced to the two-dimensional map (24), it is sure that the existence of codimension-2 Hopf bifurcation of the vibro-impact system in Figure 1 can be discussed through applying the lemma mentioned below to analyze the existence of Hopf bifurcation of map $f^2(\varepsilon; X)$.

Lemma (Iooss [13]). *Let F_ε be a one-parameter family of diffeomorphisms on R^2, satisfying the following conditions*:

H1. $F_\varepsilon(\varepsilon, 0) = 0$ *for all* ε;
H2. $D_X F_\varepsilon\,(0, 0)$ *has two conjugated eigenvalues* $\lambda_0, \bar{\lambda}_0$ *with* $|\lambda_0| = |\bar{\lambda}_0| = 1$

H3. $\left.\dfrac{\mathrm{d}|\lambda(\varepsilon)|}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} > 0$

H4. $\lambda^m(0) \neq 1, \quad m = 1, 2, 3, 4, 5.$

Subject to these assumptions H1–H4, we can make smooth ε-dependent change of the co-ordinate bringing F_ε into the form

$$F_\varepsilon(x_1, x_2) = NF_\varepsilon(x_1, x_2) + 0(|X|^5) \tag{29}$$

in polar co-ordinates

$$NF_\varepsilon = (|\lambda(\varepsilon)|r - f_1(\varepsilon)r^3,\ \phi + \theta(\varepsilon) + f_3(\varepsilon)r^3). \tag{30}$$

If $f_1(0) > 0\,(f_1(0) < 0)$, F_ε has an attracting (repelling) invariant circle for $\varepsilon > 0$ ($\varepsilon < 0$). Suppose that the complex form of F_0 is

$$F_0(z) = \lambda_0 z + \sum_{i+j=2}^{3} g_{ij}(0)\frac{z^i \bar{z}^j}{i!\,j!} + O(|z|^4), \tag{31}$$

then

$$f_1(0) = \mathrm{Re}\left[\frac{(1 - 2\lambda_0)\bar{\lambda}_0}{2(1 - \lambda_0)} g_{20}g_{11}\right] + \frac{1}{2}|g_{11}|^2 + \frac{1}{4}|g_{02}|^2 - \mathrm{Re}\left(\frac{\bar{\lambda}_0 g_{21}}{2}\right) \tag{32}$$

where $\lambda_0 = \lambda(0)$ and $g_{ij} = g_{ij}(0)$.

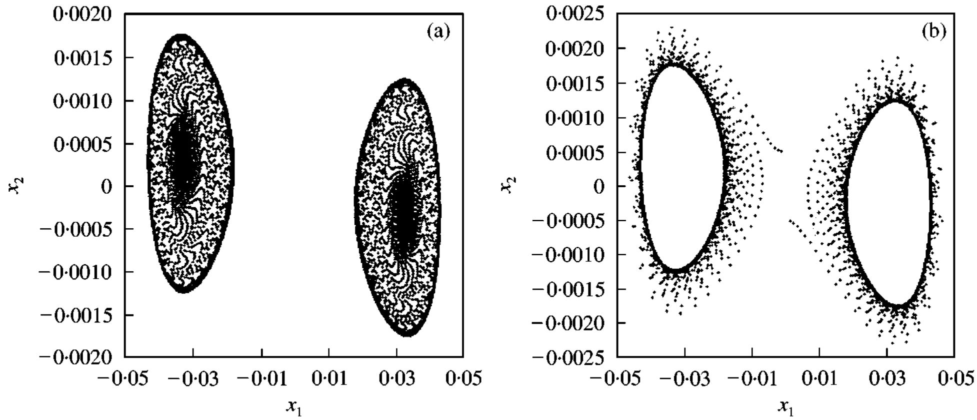

Figure 2. The codimension-2 Hopf circles (attracting inside and outside) of the normal form map (23): $\varepsilon_1 = 0{\cdot}010146$, $\varepsilon_2 = 0{\cdot}012498$, $a_1 = 7{\cdot}628514$ and $a_2 = -6{\cdot}951389$.

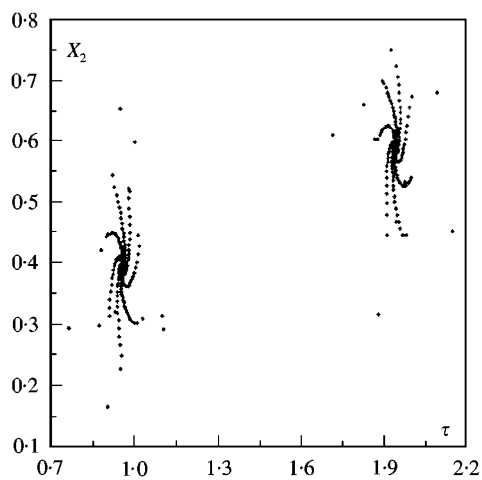

Figure 3. The stable period 2 points corresponding to the periodic 2-2 impact motion of the vibro-impact system shown in projected Poincaré section: $\mu_m = 6{\cdot}161868$, $\mu_k = 2$, $\varsigma = 0$, $f_2 = 0$, $R = 0{\cdot}8$, $b = 1{\cdot}17937$ and $v = 0{\cdot}764$.

3. NUMERICAL SIMULATIONS OF CODIMENSION-2 HOPF BIFURCATION

Let $\mu_m = 6{\cdot}161868$, $\mu_k = 2$, $\varsigma = 0$, $f_2 = 0$, $R = 0{\cdot}8$ in the two-degree-of-freedom vibro-impact system in Figure 1. According to the critical value of bifurcation parameter which was obtained from the theoretical analysis in the former section, as shown:

$$v_c = 0{\cdot}76104, \quad b_c = 1{\cdot}17359, \quad a_1 = 7{\cdot}628514, \quad a_2 = -6{\cdot}951389, \tag{33}$$

$$\varepsilon_1 = 0{\cdot}080339\mu_1 + 1{\cdot}630813\mu_2, \; \varepsilon_2 = -0{\cdot}080339\mu_1 + 2{\cdot}286738\,\mu_2, \tag{34}$$

we can find that while $\varepsilon_1^* = 0{\cdot}004925$, $\varepsilon_2 = \varepsilon_2^* = 0{\cdot}005457$ (i.e., $\varepsilon = 0$), a supercritical Hopf bifurcation takes place for the map $f^2(\varepsilon; X)$ satisfying those conditions in the lemma presented in section 2; while $\varepsilon_2 > \varepsilon_2^*$ (i.e., $\varepsilon > 0$) and $|\varepsilon_2 - \varepsilon_2^*|$ is sufficiently small, there exists an attracting Hopf circle (attracting inside and outside) for the map $f^2(\varepsilon; X)$ (i.e., stable

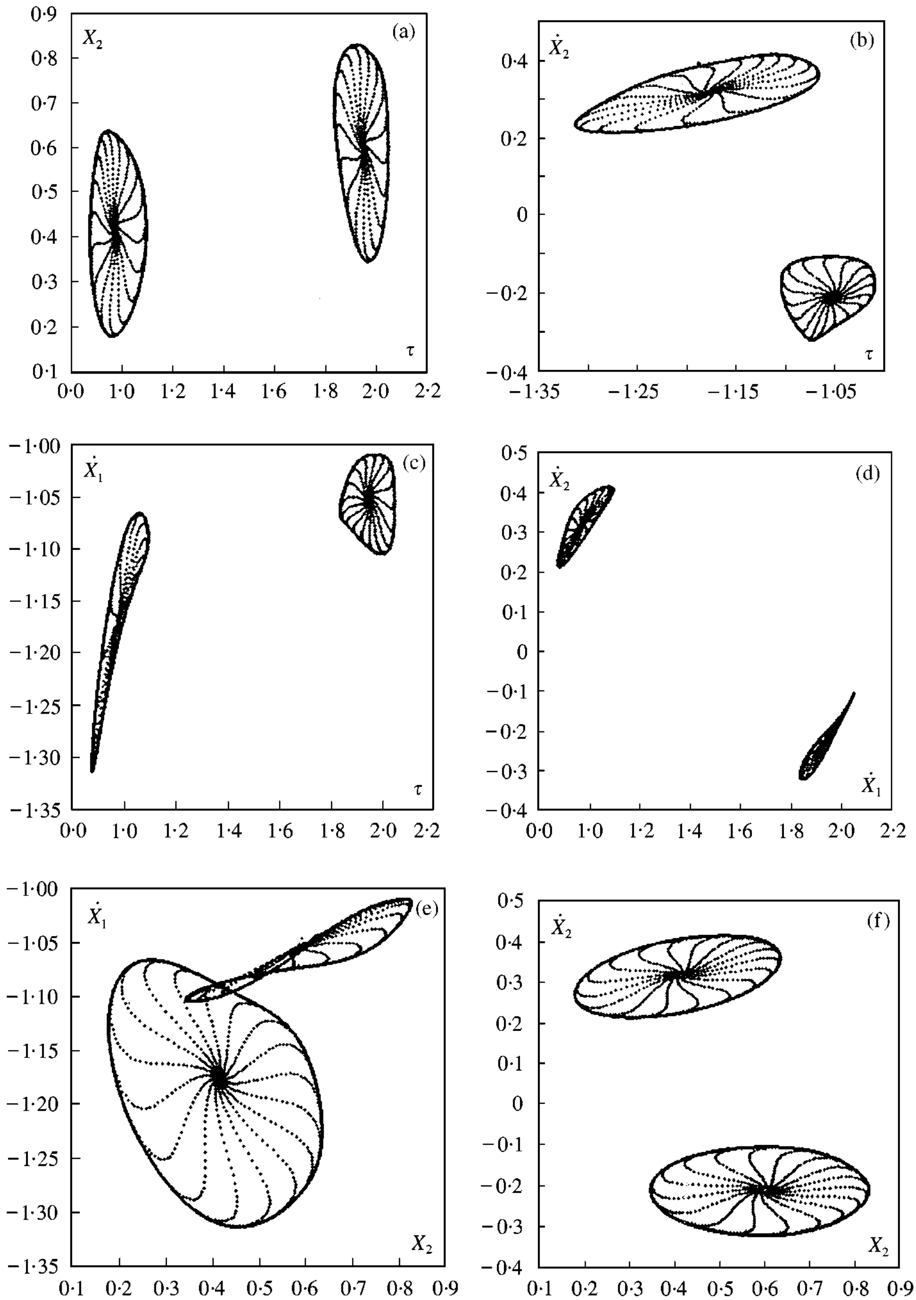

Figure 4. The stable codimension-2 Hopf circles (quasi-periodic motion of the vibro-impact system) shown in projected Poincaré sections: $\mu_m = 6{\cdot}161868$, $\mu_k = 2$, $\varsigma = 0, f_2 = 0$, $R = 0{\cdot}8$, $b = 1{\cdot}17937$ and $v = 0{\cdot}77$.

codimension-2 Hopf circle of the normal form map (23) (Figure 2)), which corresponds to the codimension-2 Hopf circles of the original Poincaré map (8) of the vibro-impact system (1). Hence, one can easily finish the work of numerical simulations on the codimension-2 Hopf bifurcation of the four-dimensional Poincaré map (8) according to the critical value of bifurcation parameter obtained from the theoretical analysis cited before, as shown in Figures 4a–f, where $b = 1{\cdot}17937$, $v = 0{\cdot}77$ (i.e., $\varepsilon_1 = 0{\cdot}010146$, $\varepsilon_2 = 0{\cdot}012498$; the

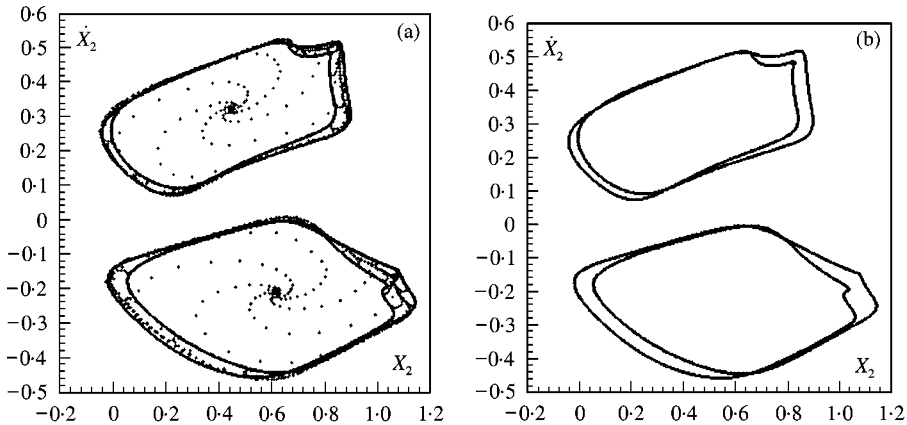

Figure 5. The codimension-2 torus doubling of the vibro-impact system shown in projected Poincaré sections: $\mu_m = 6{\cdot}161868$, $\mu_k = 2$, $\varsigma = 0, f_2 = 0$, $R = 0{\cdot}8$, $b = 1{\cdot}17937$ and $v = 0{\cdot}7785$, where the result in (b) is obtained by ignoring the first 1000 impacts among the 8000 impacts in (a).

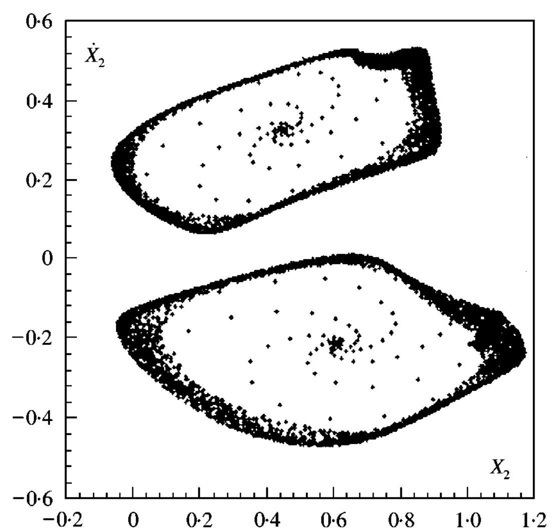

Figure 6. The choatic motion of the vibro-impact system shown in projected Poincaré section: $\mu_m = 6{\cdot}161868$, $\mu_k = 2$, $\varsigma = 0, f_2 = 0$, $R = 0{\cdot}8$, $b = 1{\cdot}17937$ and $v = 0{\cdot}7793$.

corresponding plot of the normal form map (23) is shown in Figure 2). These codimension-2 Hopf circles in the projected Poincaré sections represent the quasi-periodic impact response of the system in Figure 1. As the value of v moves further away from the one for codimension-2 Hopf bifurcation, an observable codimension-2 torus-doubling bifurcation plotted in Figure 5a, b occurs. Following the single codimension-2 torus-doubling bifurcation, the system settles into chaotic motion as shown in Figure 6.

4. CONCLUSIONS AND DISCUSSION OF RESULTS

In this paper, we have studied the codimension-2 quasi-periodic impacts of the system shown in Figure 1 by theoretical analysis and numerical simulations. It is certain that there exists codimension-2 Hopf bifurcations in vibro-impacting system with two or more degrees of freedom under suitable system parameters. The method stated in section 2 is effective for other vibro-impacting models with two degrees of freedom to conclude the existence of codimension-2 Hopf bifurcation of them. However, due to the specific local property of the center manifold, the corresponding 2-dimensional normal forms (23) fails to analyze the codimension-2 torus-doubling bifurcation, which is necessary to make a further theoretical study.

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REFERENCES

1. S. W. SHAW and P. J. HOMES 1983 *Journal of Applied Mechanics* **50**, 894–857. A Periodically forced impact oscillator with large dissipation.
2. G. S. W. HISTON 1987 *Journal of Sound and Vibration* **118**, 395–429. Global dynamics vibro-impact oscillator.
3. A. B. NORDMARK 1991 *Journal of Sound and Vibration* **145**, 279–297. Non-periodic motion caused by grazing incidence in an impact oscillator.
4. G. S. WHISTON 1992 *Journal of Sound and Vibration* **152**, 427–460. Singularities in vibro-impact dynamics.
5. C. BUDD and F. DUX 1995 *Journal of Sound and Vibration* **184**, 475–502. The effect of frequency and clearance variations on single-degree-of-freedom impact oscillators.
6. J. H. XIE 1996 *Applied Mathematics and Mechanics* (in China) **17**, 65–75. Codimension two bifurcations and Hopf bifurcations of an impacting vibrating system.
7. A. P. IVANON 1993 *Journal of Sound and Vibration* **178**, 361–378. Impact oscillations: linear theory of stability and bifurcations.
8. G. W. LUO and J. H. XIE 1998 *Journal of Sound and Vibration* **213**, 391–408. Hopf bifurcation of a two-degree-of-freedom vibrato impact system.
9. G. W. LUO J. H. XIE and X. F. SUN 1999 *Acta Mechanica Solida Sinica* **12**, 279–282. Quasi-periodic and chaotic behaviour of a two-degree-of-freedom impact system in a strong resonance case.
10. J. CAR 1981 Applied Mathematical Sciences Vol. **35**, Berlin: Springer 33–36. *Applications of Center Manifold Theory*.
11. G. IOOSS 1987 Universite de Nice. *Forms normales d'applications caractérisation globale et méthode de calcul*.
12. C. ELPHICK, E. TIRAPEGUI, M. BRACHET, P. COULLET and G. IOOSS 1987 *Physica D* **29**, 95–127. A simple global characterization for normal forms of singular vector fields.
13. G. IOOSS 1979 Mathematics Studies Vol. **36**, Amsterdam: North-Holland. Bifurcation of maps and applications.
14. G. IOOSS and D. D. JOSEPH 1990 *Elementary stability and bifurcation theory*, Berlin: Springer. pp. 43–58.
15. V. I. ARNOLD 1982 *Geometrical methods in the theory of ordina, y differential equations*. Berlin: Springer.
16. K. KUMIHIKO 1983 *Progress of Theoretical Physics* **69**, 1806–1810. Doubling of torus.
17. Y. H. WAN 1978 *SIAM*, *Journal of Applied Mathematics* **34**, 167–175. Computation of the stability condition for the Hopf bifurcation of diffeomorphism on R^2.
18. J. GUCKENHEIMER and P. J. HOLMES 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*. New York: Springer.